

LOCAL ASYMPTOTIC NORMALITY IN δ -NEIGHBORHOODS OF STANDARD GENERALIZED PARETO PROCESSES

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ABSTRACT. De Haan and Pereira (2006) provided models for spatial extremes in the case of stationarity, which depend on just one parameter $\beta > 0$ measuring tail dependence, and they proposed different estimators for this parameter. This framework was supplemented in Falk (2011) by establishing local asymptotic normality (LAN) of a corresponding point process of exceedances above a high multivariate threshold, yielding in particular asymptotic efficient estimators.

The estimators investigated in these papers are based on a finite set of points t_1, \dots, t_d , at which observations are taken. We generalize this approach in the context of functional extreme value theory (EVT). This more general framework allows estimation over some spatial parameter space, i.e., the finite set of points t_1, \dots, t_d is replaced by $t \in [a, b]$. In particular, we derive efficient estimators of β based on those processes in a sample of iid processes in $C[0, 1]$ which exceed a given threshold function.

1. INTRODUCTION

Suppose that the stochastic process $\mathbf{V} = (V_t)_{t \in [0, 1]} \in C[0, 1]$ is a *standard generalized Pareto process* (GPP) (Buishand et al. (2008)), i.e., there exists $x_0 > 0$ such that

$$P(\mathbf{V} \leq f) = 1 + \log(G(f)), \quad f \in \bar{E}^- [0, 1], \|f\|_\infty \leq x_0,$$

where $\bar{E}^- [0, 1]$ is the set of those bounded functions on $[0, 1]$ that attain only nonpositive values and which have a finite set of discontinuities. By G we denote the functional distribution function (df) of a *standard extreme value process* (EVP) $\boldsymbol{\eta} = (\eta_t)_{t \in [0, 1]} \in C[0, 1]$, i.e.,

$$G(f) = P(\boldsymbol{\eta} \leq f), \quad f \in \bar{E}^- [0, 1],$$

$P(\eta_t \leq x) = \exp(x)$, $x \leq 0$, $t \in [0, 1]$, and $\boldsymbol{\eta}$ is *max-stable*:

$$P\left(\boldsymbol{\eta} \leq \frac{f}{n}\right)^n = P(\boldsymbol{\eta} \leq f), \quad f \in \bar{E}^- [0, 1], n \in \mathbb{N}.$$

All operations on functions such as \leq , multiplication with a constant etc. are meant componentwise. For random functions, i.e., stochastic processes such as \mathbf{V} , $\boldsymbol{\eta}$ we use bold letters, to distinguish these from nonrandom functions such as f .

De Haan and Pereira (2006) provided models for spatial extremes in the case of stationarity, which depend on just one parameter $\beta > 0$ measuring tail dependence, and they proposed different estimators for this parameter. This framework was supplemented in Falk (2011) by establishing local asymptotic normality (LAN) of a corresponding point process of exceedances above a high multivariate threshold.

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Precisely, it is assumed that for any $x_1, \dots, x_d \leq 0$, $d \in \mathbb{N}$,

$$(1) \quad P(\eta_{t_j} \leq x_j, 1 \leq j \leq d) = \exp \left(- \int_{-\infty}^{\infty} \max_{j \leq d} |x_j| \psi_{\beta}(s - t_j) ds \right),$$

where $\psi_{\beta}(s) = \beta \psi(\beta s)$ with a scale parameter $\beta > 0$, and ψ is a continuous probability density on \mathbb{R} with $\psi(s) = \psi(-s) > 0$ and $\psi(s)$, $s \geq 0$, decreasing.

In the papers by de Haan and Pereira (2006) and Falk (2011) the density ψ is known and the parameter β is estimated. The estimators investigated in these papers are based on a finite set of points $t_1 < \dots < t_d$; estimation over some interval $t \in [a, b]$ seems to be an open problem. This is the content of the present paper, which is organized as follows. In Section 2 we compile some auxiliary results and tools, in particular from functional extreme value theory (EVT). In Section 3 we introduce our estimator of β and establish its asymptotic normality under the condition that the underlying observations $\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(n)}$ are independent copies of a standard GPP \mathbf{V} . Local asymptotic normality (LAN) of a corresponding point process of exceedances above a high constant threshold function is established in Section 4. This is achieved under the condition that the underlying observations are in a δ -neighborhood of a standard GPP. As an application we obtain from LAN-theory that our estimator of β is asymptotically efficient in this setup. For an account of functional EVT we refer to de Haan and Ferreira (2006); for a supplement including in particular basics of GPP we refer to Aulbach et al. (2011).

2. AUXILIARY RESULTS AND TOOLS

In this section we compile several auxiliary results and tools. We start with the functional df of a standard EVP $\boldsymbol{\eta} \in C[0, 1]$, whose finite dimensional marginal distributions (fidis) are given by equation (1).

Lemma 2.1. *We have for any $f \in \bar{E}^-[0, 1]$*

$$P(\boldsymbol{\eta} \leq f) = \exp \left(- \int_{-\infty}^{\infty} \sup_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) ds \right).$$

Proof. The assertion follows from the fact that a probability measure is continuous from above together with the dominated convergence theorem; note that $\int_{-\infty}^{\infty} \sup_{t \in [0, 1]} \psi(s - \beta t) ds < \infty$. Let $Q = \{q_1, q_2, \dots\}$ be a denumerable and dense subset of $[0, 1]$, which contains also the set of discontinuities of f . Recall that $\boldsymbol{\eta} \in C[0, 1]$. From representation (1) we obtain

$$\begin{aligned} P(\boldsymbol{\eta} \leq f) &= P \left(\bigcap_{n \in \mathbb{N}} \{ \eta_{q_i} \leq f(q_i), 1 \leq i \leq n \} \right) \\ &= \lim_{n \rightarrow \infty} P(\eta_{q_i} \leq f(q_i), 1 \leq i \leq n) \\ &= \lim_{n \rightarrow \infty} \exp \left(- \int_{-\infty}^{\infty} \max_{1 \leq i \leq n} (|f(q_i)| \psi_{\beta}(s - q_i)) ds \right) \\ &= \exp \left(- \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} \left(\max_{1 \leq i \leq n} (|f(q_i)| \psi_{\beta}(s - q_i)) \right) ds \right) \\ &= \exp \left(- \int_{-\infty}^{\infty} \sup_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) ds \right). \end{aligned}$$

□

The preceding result provides the functional df $P(\mathbf{V} \leq f) = 1 + \log(G(f))$ of the GPP \mathbf{V} in its upper tail.

Corollary 2.2. *There exists $x_0 > 0$ such that for the GPP \mathbf{V} corresponding to the EVP $\boldsymbol{\eta}$ and for any $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq x_0$*

(i)

$$P(\mathbf{V} \leq f) = 1 - \int_{-\infty}^{\infty} \sup_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) \, ds,$$

(ii)

$$P(\mathbf{V} > f) = \int_{-\infty}^{\infty} \inf_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) \, ds.$$

Proof. While part (i) is an immediate consequence of Lemma 2.1, part (ii) follows from the inclusion-exclusion formula as in the proof of Lemma 3.1 in Falk (2011). \square

Note that

$$\|f\|_D := \int_{-\infty}^{\infty} \sup_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) \, ds,$$

defines a norm on the set $E[0, 1]$, called *D-norm*. By $E[0, 1]$ we denote the set of those functions on $[0, 1]$, which are bounded and have a finite number of discontinuities. The representation of a multivariate extreme value distribution (EVD) or of a multivariate generalized Pareto distribution (GPD) in terms of a *D-norm* is well-known, see Falk et al. (2010). This concept was extended to functional spaces in Aulbach et al. (2011). The fids of the stochastic processes $\boldsymbol{\eta}$ or \mathbf{V} are obtained by considering the function $f(t) = \sum_{i=1}^d x_i 1_{\{t_i\}}(t) \in \bar{E}^-[0, 1]$, $x_i \leq 0$, $t_i \in [0, 1]$, $d \in \mathbb{N}$. This norm satisfies, for example, the general inequality

$$\|f\|_\infty \leq \|f\|_D \leq \|f\|_\infty \|1\|_D, \quad f \in E[0, 1],$$

where 1 denotes the constant function one and $\|f\|_\infty := \sup_{t \in [0, 1]} |f(t)|$ is the usual sup-norm. This inequality implies in particular that each *D-norm* is equivalent with the sup-norm which, in turn, implies that the L_p -norm $\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt \right)^{1/p}$, with $p \in [1, \infty)$, is not a *D-norm*.

The following auxiliary result is a crucial tool for the derivation of estimators of β .

Lemma 2.3. *We have*

$$\int_{-\infty}^{\infty} \inf_{t \in [0, 1]} \psi(s - \beta t) \, ds = 2 \left(1 - \Psi \left(\frac{\beta}{2} \right) \right) = 2\Psi \left(-\frac{\beta}{2} \right),$$

where $\Psi(x) = \int_{-\infty}^x \psi(s) \, ds$.

Proof. We have

$$\begin{aligned} \int_{-\infty}^{\infty} \inf_{t \in [0, 1]} \psi(s - \beta t) \, ds &= \int_{-\infty}^{\infty} \inf_{t \in [0, 1]} \psi(|s - \beta t|) \, ds \\ &= \int_{-\infty}^{\infty} \min(\psi(|s|), \psi(|s - \beta|)) \, ds \\ &= \int_{\beta/2}^{\infty} \psi(s) \, ds + \int_{-\infty}^{\beta/2} \psi(s - \beta) \, ds \\ &= 2\Psi(-\beta/2). \end{aligned}$$

\square

3. ESTIMATION OF β

A natural estimator of $\Psi(-\beta/2)$, based on independent copies $\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(n)}$ of \mathbf{V} , is by Corollary 2.2 and Lemma 2.3 given by

$$\hat{\Psi}_{c,n} := \frac{1}{2|c|n} \sum_{i=1}^n 1_{(c,0]}(\mathbf{V}^{(i)}).$$

Note the twofold meaning of c : In the denominator $2|c|n$ this is just the absolute value of the constant $c < 0$, whereas in the term $1_{(c,0]}(\mathbf{V}^{(i)})$ we mean the constant *function* c , and we have $1_{(c,0]}(\mathbf{V}^{(i)}) = 1$ if and only if each component satisfies $V_t^{(i)} > c$, $t \in [0, 1]$. There should be no risk of confusion.

The law of large number implies

$$\hat{\Psi}_{c,n} \xrightarrow{n \rightarrow \infty} \Psi\left(-\frac{\beta}{2}\right) \quad \text{a.s.}$$

and, thus,

$$\hat{\beta}_{c,n} := -2\Psi^{-1}\left(\hat{\Psi}_{c,n}\right) \xrightarrow{n \rightarrow \infty} \beta \quad \text{a.s.},$$

where $F^{-1}(q) := \inf\{t \in \mathbb{R} : F(t) \geq q\}$, $q \in (0, 1)$, denotes the generalized inverse of a df F .

The Moivre-Laplace theorem implies asymptotic normality of $\hat{\Psi}_{c,n}$ and $\hat{\beta}_{c,n}$, i.e., the next result is a functional counterpart of Proposition 3.3 in Falk (2011).

Proposition 3.1. *For $c < 0$ close enough to 0 we have*

$$\begin{aligned} & n^{1/2} \left(\hat{\Psi}_{c,n} - \Psi\left(-\frac{\beta}{2}\right) \right) \\ & \rightarrow_D N \left(0, \frac{\Psi\left(-\frac{\beta}{2}\right) \left(1 - 2|c| \Psi\left(-\frac{\beta}{2}\right)\right)}{2|c|} \right) \end{aligned}$$

and

$$n^{1/2} \left(\hat{\beta}_{c,n} - \beta \right) \rightarrow_D N \left(0, \frac{2\Psi\left(-\frac{\beta}{2}\right) \left(1 - 2|c| \Psi\left(-\frac{\beta}{2}\right)\right)}{|c| \psi^2\left(-\frac{\beta}{2}\right)} \right).$$

We now consider a stochastic process $\mathbf{X} \in \bar{C}^-[0, 1] := \{f \in C[0, 1] : f \leq 0\}$, whose upper tail is in a δ -neighborhood of that of a GPP $\mathbf{V} \in C^-[0, 1]$ with D -norm $\|\mathbf{f}\|_D = \int_{-\infty}^{\infty} \sup_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) ds$. Precisely, we require that

$$(C) \quad P(\mathbf{X} > cf) = P(\mathbf{V} > cf) (1 + c^\delta K(f) + r(c, f))$$

for $c \in (0, 1)$ and $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq \varepsilon_0$ for some $\varepsilon_0 > 0$, where $K : \bar{E}^-[0, 1] \rightarrow \mathbb{R}$ is a function and the remainder $r(c, f)$ is of order $o(c^\delta)$ as $c \rightarrow 0$. The next result is an immediate consequence of Corollary 2.2.

Lemma 3.2. *Suppose that the stochastic process $\mathbf{X} \in \bar{C}^-[0, 1]$ satisfies condition (C). Then we obtain for $c \in (0, 1)$ and $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq \varepsilon_0$*

$$P(\mathbf{X} > cf) = c \left(\int_{-\infty}^{\infty} \inf_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) ds \right) (1 + c^\delta K(f) + r(c, f)).$$

In what follows we show how a process \mathbf{X} satisfying condition (C) can be generated. From Aulbach et al. (2011) we conclude that there is a stochastic process $\mathbf{Z} = (Z_t)_{t \in [0,1]}$ on $[0, 1]$ with continuous sample paths and $0 \leq Z_t \leq m$, $E(Z_t) = 1$, $t \in [0, 1]$, for some constant $m \geq 1$, such that

$$\begin{aligned} \|f\|_D &= \int_{-\infty}^{\infty} \sup_{t \in [0,1]} (|f(t)| \psi(s - \beta t)) ds \\ &= E \left(\sup_{t \in [0,1]} (|f(t)| Z_t) \right), \quad f \in E[0, 1]. \end{aligned}$$

The stochastic process \mathbf{Z} is called *generator* of the D -norm. Conversely, each process \mathbf{Z} with the above properties generates a D -norm via $\|f\|_D := E \left(\sup_{t \in [0,1]} (|f(t)| Z_t) \right)$, $f \in E[0, 1]$. For every D -norm $\|\cdot\|_D$ there exists a standard EVP $\boldsymbol{\eta} \in C[0, 1]$ with functional df $P(\boldsymbol{\eta} \leq f) = \exp(-\|f\|_D)$, $f \in \bar{E}^-[0, 1]$. While a generator \mathbf{Z} is in general not uniquely determined, the *generator constant* $E \left(\sup_{t \in [0,1]} (Z_t) \right) = \|1\|_D$ is. We refer to Aulbach et al. (2011) for details.

Put

$$(2) \quad \mathbf{V} := (V_t)_{t \in [0,1]} := \left(\max \left(-\frac{U}{Z_t}, M \right) \right)_{t \in [0,1]},$$

where U and \mathbf{Z} are independent, U is a uniformly on $(0, 1)$ distributed rv and $M < 0$ is an arbitrary constant. We incorporate the constant M to ensure that $V_t > -\infty$ for each $t \in [0, 1]$, as Z_t may attain the value zero. The continuous process \mathbf{V} is a GPP, as we have for $f \in \bar{E}^-[0, 1]$ with $\|f\|_\infty \leq \min(|M|, 1/m)$

$$\begin{aligned} P(\mathbf{V} \leq f) &= P(U \geq |f(t)| Z_t, 0 \leq t \leq 1) \\ &= P(U \geq \sup_{t \in [0,1]} (|f(t)| Z_t)) \\ &= 1 - E \left(\sup_{t \in [0,1]} (|f(t)| Z_t) \right) \\ &= 1 - \|f\|_D. \end{aligned}$$

We have, moreover,

$$\begin{aligned} P(\mathbf{V} > f) &= P(U < |f(t)| Z_t, 0 \leq t \leq 1) \\ &= P(U \leq \inf_{t \in [0,1]} (|f(t)| Z_t)) \\ &= E \left(\inf_{t \in [0,1]} (|f(t)| Z_t) \right) \\ &= \int_{-\infty}^{\infty} \inf_{t \in [0,1]} (|f(t)| \psi(s - \beta t)) ds, \end{aligned}$$

where the final equality is a consequence of Corollary 2.2, part (ii).

Replace now the rv U in (2) by a rv $Y > 0$, which is also independent of \mathbf{Z} and whose df H is continuous and satisfies

$$(3) \quad H(u) = u + Au^{1+\delta} + o(u^{1+\delta}) \quad \text{as } u \downarrow 0$$

with some constant $A \in \mathbb{R}$. The standard exponential distribution, for instance, satisfies this condition with $\delta = 1$ and $A = -1/2$. The process

$$(4) \quad \mathbf{X} := (X_t)_{t \in [0,1]} := \left(\max \left(-\frac{Y}{Z_t}, M \right) \right)_{t \in [0,1]}$$

then satisfies condition (C) with

$$K(f) = A \frac{E \left(\inf_{t \in [0,1]} (|f(t)| Z_t)^{1+\delta} \right)}{E \left(\inf_{t \in [0,1]} (|f(t)| Z_t) \right)},$$

which has to be interpreted as zero if the denominator vanishes.

The following theorem is the main result of this section. We will see in Section 4 using LAN theory that it implies that $\hat{\Psi}_{c_n, n}$ is an asymptotically efficient estimator sequence in an appropriate model.

Theorem 3.3. *Suppose that the stochastic process $\mathbf{X} \in \bar{C}^- [0, 1]$ satisfies condition (C). If the sequence of thresholds $c_n < 0$, $n \in \mathbb{N}$, satisfies $c_n \rightarrow 0$, $n |c_n| \rightarrow \infty$, $n |c_n|^{1+2\delta} \rightarrow \text{const} \geq 0$ as $n \rightarrow \infty$, then we obtain*

(i)

$$(n |c_n|)^{1/2} \left(\hat{\Psi}_{c_n, n} - \Psi \left(-\frac{\beta}{2} \right) \right) \rightarrow_D N \left(\text{const}^{1/2} \mu, \frac{1}{2} \Psi \left(-\frac{\beta}{2} \right) \right),$$

(ii)

$$(n |c_n|)^{1/2} \left(\hat{\beta}_{c_n, n} - \beta \right) \rightarrow_D N \left(-\frac{2 \text{const}^{1/2} \mu}{\psi \left(-\frac{\beta}{2} \right)}, \frac{2 \Psi \left(-\frac{\beta}{2} \right)}{\psi^2 \left(-\frac{\beta}{2} \right)} \right),$$

where $\mu := K(-1) \Psi(-\beta/2)$.

Proof. From Lemma 3.2 we obtain

$$\begin{aligned} & \frac{1}{|c|^\delta} \left(\frac{P(\mathbf{X} > c)}{|c|} - \int_{-\infty}^{\infty} \inf_{t \in [0,1]} \psi(s - \beta t) ds \right) \\ & \rightarrow_{c \uparrow 0} K(-1) \int_{-\infty}^{\infty} \inf_{t \in [0,1]} \psi(s - \beta t) ds \\ & = 2K(-1) \Psi \left(-\frac{\beta}{2} \right). \end{aligned} \tag{5}$$

Write

$$\begin{aligned} & (n |c_n|)^{1/2} \left(\hat{\Psi}_{c_n, n} - \Psi \left(-\frac{\beta}{2} \right) \right) \\ & = (n |c_n|)^{1/2} \left(\frac{1}{2n |c_n|} \sum_{j=1}^n (1_{(c_n, 0]}(\mathbf{X}_j) - P(\mathbf{X} > c_n)) \right) \\ & \quad + (n |c_n|)^{1/2} \left(\frac{P(\mathbf{X} > c_n)}{2 |c_n|} - \Psi \left(-\frac{\beta}{2} \right) \right) \\ & =: \eta_n + b_n. \end{aligned}$$

The Moivre-Laplace theorem implies

$$\eta_n \rightarrow_D N \left(0, \frac{1}{2} \Psi \left(-\frac{\beta}{2} \right) \right),$$

and expansion (5) yields

$$\begin{aligned} b_n & = \frac{(n |c_n|^{1+2\delta})^{1/2}}{2} \frac{1}{|c_n|^\delta} \left(\frac{P(\mathbf{X} > c_n)}{|c_n|} - \int_{-\infty}^{\infty} \inf_{t \in [0,1]} \psi(s - \beta t) ds \right) \\ & \rightarrow_{n \rightarrow \infty} \text{const}^{1/2} \mu. \end{aligned}$$

Equally, one concludes

$$\begin{aligned}
& (n |c_n|)^{1/2} \left(\hat{\beta}_{c_n, n} - \beta \right) \\
&= 2(n |c_n|)^{1/2} \left(\Psi^{-1} \left(\Psi \left(-\frac{\beta}{2} \right) \right) - \Psi^{-1} \left(\hat{\Psi}_{c_n, n} \right) \right) \\
&= 2(n |c_n|)^{1/2} (\Psi^{-1})'(\xi) \left(\Psi \left(-\frac{\beta}{2} \right) - \hat{\Psi}_{c_n, n} \right) \\
&\rightarrow_D N \left(-\frac{2 \text{const}^{1/2} \mu}{\psi \left(-\frac{\beta}{2} \right)}, \frac{2 \Psi \left(-\frac{\beta}{2} \right)}{\psi^2 \left(-\frac{\beta}{2} \right)} \right)
\end{aligned}$$

by Slutsky's lemma, with ξ between $\hat{\Psi}_{c_n, n}$ and $\Psi(-\beta/2)$. This completes the proof. \square

The idea suggests itself to substitute the constant threshold c by a suitable threshold function $f \in \bar{E}^- [0, 1]$ and to consider, with $c < 0$,

$$\begin{aligned}
\hat{\Psi}_{f, c, n} &:= \frac{1}{2 |c| n} \sum_{i=1}^n 1(\mathbf{V}^{(i)} > |c| f) \\
&\rightarrow_{n \rightarrow \infty} \frac{1}{2 |c|} P(\mathbf{V} > |c| f) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \inf_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) ds
\end{aligned}$$

almost surely by the law of large numbers and Corollary 2.2.

The fact that with constant function $f = -1$, the above integral equals by Lemma 2.3

$$\int_{-\infty}^{\infty} \inf_{t \in [0, 1]} \psi(s - \beta t) ds = 2 \Psi \left(-\frac{\beta}{2} \right)$$

was the crucial observation for the derivation of an estimator of β . Substituting the constant function $f = -1$ by an arbitrary function $f \in \bar{E}^- [0, 1]$ can, however, lead to surprising consequences, as the following example shows.

EXAMPLE 3.4. Take $\psi(s) = 2^{-1} \exp(-|s|)$, $s \in \mathbb{R}$, and $f(t) := -\exp(-t)$, $t \in [0, 1]$. Then we have for any $\beta \in [0, 1]$

$$T(f, \beta, \psi) := \int_{-\infty}^{\infty} \inf_{t \in [0, 1]} (|f(t)| \psi(s - \beta t)) ds = \exp(-1),$$

i.e., the functional $T(f, \beta, \psi)$ is not capable to discriminate between different values of $\beta \in [0, 1]$. For $\beta > 1$ one obtains, however,

$$T(f, \beta, \psi) = \exp \left(-\frac{1 + \beta}{2} \right).$$

The question, whether for each underlying density ψ there exists an *optimal* threshold function $f = f_\psi$, is an open problem.

4. LAN OF EXCEEDANCES

Let $\mathbf{X}^{(i)}$, $1 \leq i \leq n$, be independent copies of a stochastic process $\mathbf{X} \in \bar{C}^- [0, 1]$, which satisfies condition (C). Choose $c < 0$. In this section we establish local asymptotic normality (LAN) of the point process of exceedances

$$N_{n, c}(B) := \sum_{i=1}^n \varepsilon_{\sup_{t \in [0, 1]} (X_t^{(i)} / c)} (B \cap [0, 1)), \quad B \in \mathbb{B},$$

where \mathbb{B} denotes the σ -field of Borel sets in \mathbb{R} . Note that for $s \in (0, 1]$

$$\sup_{t \in [0, 1]} \frac{X_t}{c} < s \iff \mathbf{X} > sc,$$

i.e., the random point measure $N_{n,c}$ actually represents those processes among $\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)}$ which are exceedances above the constant function c .

It is by Theorem 3.3, part (i), quite convenient to substitute the parameter $\beta > 0$ in the family $\psi_\beta(\cdot) = \beta\psi(\beta\cdot)$ by the parameter

$$\vartheta := 2\Psi\left(-\frac{\beta}{2}\right) \in (0, 1).$$

Fix $\vartheta_0 \in (0, 1)$. We require that the family of univariate df $F_{\vartheta,c}(s) := P_\vartheta(\mathbf{X} > sc)$, $\vartheta \in (0, 1)$, $s > 0$, satisfies for $s \in (0, 1)$, $c_0 \leq c < 0$ for some $c_0 < 0$, and ϑ close to ϑ_0 the expansion

$$(D) \quad \frac{f_{\vartheta,c}(s)}{f_{\vartheta_0,c}(s)} := \frac{\frac{d}{ds}P_\vartheta(\mathbf{X} > sc)}{\frac{d}{ds}P_{\vartheta_0}(\mathbf{X} > sc)} = 1 + L(\vartheta - \vartheta_0) + r_{\vartheta_0}(s, \vartheta, c),$$

with

$$r_{\vartheta_0}(s, \vartheta, c) = o(|\vartheta - \vartheta_0|) + O(|c|^\gamma)$$

uniformly for $s \in (0, 1)$, $c_0 \leq c < 0$ and ϑ close to ϑ_0 , where the constants $L \in \mathbb{R}$ and $\gamma > 0$ may depend on ϑ_0 . Note that condition (D) implies in particular that $P_\vartheta\left(\sup_{t \in [0, 1]} (X_t/c) = s\right) = 0$ and, thus, $F_{\vartheta,c}(s) = P_\vartheta\left(\sup_{t \in [0, 1]} (X_t/c) \leq s\right)$, $s > 0$, is actually a df on $[0, \infty)$.

Condition (D) is, for example, satisfied with $L = 1/\vartheta_0$ and $r_{\vartheta_0} = 0$ if \mathbf{X} is a GPP. We can also use the approach from definition (4) to generate a process

$$(6) \quad \mathbf{X} := \left(\max\left(-\frac{Y}{Z_t}, M\right) \right)_{t \in [0, 1]}.$$

In addition to condition (3) we require that the continuous df H of the rv $Y > 0$ satisfies the expansion

$$H(u) = u + Au^{1+\delta} + r(u), \quad 0 < u < 1,$$

with some constant $A \in \mathbb{R}$, where the function r is differentiable on $(0, 1)$ with bounded derivative and $r'(u) = o(u^\delta)$ as $u \downarrow 0$. Then condition (D) is satisfied with $L = 1/\vartheta_0$.

Denote by $Y_1, \dots, Y_{\tau(n)}$ those rv among $\sup_{t \in [0, 1]} X_t^{(i)}/c$, $1 \leq i \leq n$, with $\sup_{t \in [0, 1]} X_t^{(i)}/c < 1$, in the order of their outcome. Then we have

$$N_{n,c}(B) = \sum_{k \leq \tau(n)} \varepsilon_{Y_k}(B), \quad B \in \mathbb{B}.$$

From Theorem 1.4.1 in Reiss (1993) we obtain that Y_1, Y_2, \dots are independent copies of a rv Y with df

$$P_\vartheta(Y \leq t) = \frac{P_\vartheta(\mathbf{X} > tc)}{P_\vartheta(\mathbf{X} > c)}, \quad 0 \leq t \leq 1,$$

under parameter $\vartheta > 0$, and they are independent of the total number $\tau(n)$, which is binomial $B(n, P_\vartheta(\mathbf{X} > c))$ -distributed.

Since the distribution $\mathcal{L}_{\vartheta,c}(Y)$ of Y under ϑ is by condition (D) dominated by $\mathcal{L}_{\vartheta_0,c}(Y)$ for ϑ in a neighborhood of ϑ_0 and $c_0 \leq c < 0$, the distribution $\mathcal{L}_\vartheta(N_{n,c})$ of $N_{n,c}$ is dominated by $\mathcal{L}_{\vartheta_0}(N_{n,c})$, see, e.g. Theorem 3.1.2 in Reiss (1993). Precisely, $N_{n,c}$ is a random element in the set $\mathbb{M} := \{\mu = \sum_{1 \leq j \leq n} \varepsilon_{y_j} : y_j \geq 0, j \leq n, n = 0, 1, 2, \dots\}$ of finite point measures on $([0, \infty), \mathbb{B} \cap [0, \infty))$, equipped with the smallest σ -field \mathcal{M} such

that for any $B \in \mathbb{B} \cap [0, \infty)$ the projection $\pi_B : \mathbb{M} \rightarrow \{0, 1, 2, \dots\}$, $\pi_B(\mu) := \mu(B)$, is measurable; we refer to Section 1.1 in Reiss (1993) for technical details.

From Reiss (1993, Example 3.1.2) we conclude that $\mathcal{L}_\vartheta(N_{n,c})$ has the $\mathcal{L}_{\vartheta_0}(N_{n,c})$ -density

$$\begin{aligned} & \frac{d\mathcal{L}_\vartheta(N_{n,c})}{d\mathcal{L}_{\vartheta_0}(N_{n,c})}(\mu) \\ &= \left(\prod_{i=1}^{\mu((0,1))} \frac{f_{\vartheta,c}(y_i)}{f_{\vartheta_0,c}(y_i)} \frac{P_{\vartheta}(\mathbf{X} > c)}{P_{\vartheta_0}(\mathbf{X} > c)} \right) \\ & \quad \times \left(\frac{P_{\vartheta}(\mathbf{X} > c)}{P_{\vartheta_0}(\mathbf{X} > c)} \right)^{\mu((0,1))} \left(\frac{1 - P_{\vartheta}(\mathbf{X} > c)}{1 - P_{\vartheta_0}(\mathbf{X} > c)} \right)^{n - \mu((0,1))} \end{aligned}$$

if $\mu = \sum_{i=1}^{\mu((0,1))} \varepsilon_{y_i}$ and $\mu((0,1)) \leq n$. The loglikelihood ratio is, consequently,

$$\begin{aligned} & L_{n,c}(\vartheta \mid \vartheta_0) \\ &= \log \left\{ \frac{d\mathcal{L}_\vartheta(N_{n,c})}{d\mathcal{L}_{\vartheta_0}(N_{n,c})}(N_{n,c}) \right\} \\ &= \sum_{k \leq \tau(n)} \log \left(\frac{f_{\vartheta,c}(Y_k)}{f_{\vartheta_0,c}(Y_k)} \frac{P_{\vartheta}(\mathbf{X} > c)}{P_{\vartheta_0}(\mathbf{X} > c)} \right) \\ (7) \quad & + \tau(n) \log \left(\frac{P_{\vartheta}(\mathbf{X} > c)}{P_{\vartheta_0}(\mathbf{X} > c)} \right) + (n - \tau(n)) \log \left(\frac{1 - P_{\vartheta}(\mathbf{X} > c)}{1 - P_{\vartheta_0}(\mathbf{X} > c)} \right). \end{aligned}$$

We let in the sequel $c = c_n$ depend on the sample size n with $c_n \uparrow 0$ and, equally, $\vartheta = \vartheta_n$ with $\vartheta_n \rightarrow \vartheta_0$ as $n \rightarrow \infty$. Precisely, we put with arbitrary $\xi \in \mathbb{R}$

$$\vartheta_n := \vartheta_n(\xi) := \vartheta_0 + \frac{\xi}{(n|c_n|)^{1/2}}.$$

The following theorem is the main result of this section. It is analogous to Theorem 5.1 in Falk (2011), whose proof carries over.

Theorem 4.1. *Suppose that $\psi(s) = \psi(-s)$ and that $\psi(s)$, $s \geq 0$, is decreasing. Suppose, further, that $n|c_n| \rightarrow_{n \rightarrow \infty} \infty$ and that*

$$(8) \quad n|c_n|^{1+2\min(\delta,\gamma)} \rightarrow_{n \rightarrow \infty} 0.$$

Then we obtain the expansion

$$\begin{aligned} L_{n,c_n}(\vartheta_n \mid \vartheta_0) &= \xi LZ_n - \frac{\xi^2 L^2 \vartheta_0}{2} + o_{P_{\vartheta_0}}(1) \\ &\rightarrow_{D_{\vartheta_0}} N \left(-\frac{\xi^2 L^2 \vartheta_0}{2}, \xi^2 L^2 \vartheta_0 \right) \end{aligned}$$

with

$$(9) \quad Z_n := \frac{\tau(n) - n|c_n|\vartheta_0}{(n|c_n|)^{1/2}} \rightarrow_{D_{\vartheta_0}} N(0, \vartheta_0).$$

The above result reveals that the complete information about the underlying parameter that is contained in the exceedances $Y_1, \dots, Y_{\tau(n)}$ is, actually, contained in their number $\tau(n)$ as n increases. This is in complete accordance with the results in Falk (1998), where this phenomenon was studied for general truncated empirical processes. The result here is, however, derived under more specialized conditions.

Theorem 4.1 together with the Hajék-LeCam convolution theorem provides the asymptotically minimum variance within the classes of regular estimators of ϑ_0 . This class of estimators $\tilde{\vartheta}_n$ is defined by the property

that they are asymptotically unbiased under $\vartheta_n = \vartheta_n(\xi) = \vartheta_0 + \xi(n|c_n|)^{-1/2}$ with $\vartheta_0 \in (0, 1)$ for any $\xi \in \mathbb{R}$, precisely,

$$(n|c_n|)^{1/2} (\tilde{\vartheta}_n - \vartheta_n) \rightarrow_{D_{\vartheta_n}} Q_{\vartheta_0},$$

where the limit distribution Q_{ϑ_0} does not depend on ξ ; see, e.g. Sections 8.4 and 8.5 in Pfanzagl (1994).

By LeCam's first lemma (see, e.g., LeCam and Yang (1990, Chapter 3, Theorem 1)) we obtain that under $\vartheta_n = \vartheta_n(\xi)$

$$\begin{aligned} L_{n,c_n}(\vartheta_n | \vartheta_0) &= \xi LZ_n - \frac{\xi^2 L^2 \vartheta_0}{2} + o_{P_{\vartheta_n}}(1) \\ &\rightarrow_{D_{\vartheta_n}} N\left(\frac{\xi^2 L^2 \vartheta_0}{2}, \xi^2 L^2 \vartheta_0\right) \end{aligned}$$

with

$$(10) \quad Z_n \rightarrow_{D_{\vartheta_n}} N(\xi L \vartheta_0, \vartheta_0).$$

An efficient estimator of ϑ_0 within the class of regular estimators has necessarily the minimum limiting variance

$$\sigma_{\text{minimum}}^2 = \frac{1}{L^2 \vartheta_0},$$

which is the inverse of the limiting variance of the *central sequence* LZ_n under ϑ_0 (Pfanzagl (1994, Theorem 8.4.1)).

Consider the estimator

$$\hat{\vartheta}_n := \frac{\tau(n)}{n|c_n|}.$$

Then we have with $\vartheta_n = \vartheta_n(\xi) = \vartheta_0 + \xi(n|c_n|)^{-1/2}$

$$(n|c_n|)^{1/2} (\hat{\vartheta}_n - \vartheta_n) = (n|c_n|)^{1/2} \left(\frac{\tau(n)}{n|c_n|} - \vartheta_0 \right) - \xi = Z_n - \xi.$$

The estimator $\hat{\vartheta}_n$ is, consequently, not a regular estimator since we have by (10)

$$(n|c_n|)^{1/2} (\hat{\vartheta}_n - \vartheta_n) = Z_n - \xi \rightarrow_{D_{\vartheta_n}} N(\xi(L\vartheta_0 - 1), \vartheta_0),$$

where the limiting distribution depends on ξ unless $L = 1/\vartheta_0$. Its asymptotic relative efficiency, defined as the ratio of the limiting variances under ϑ_0 is

$$ARE(\vartheta_0) = \frac{\vartheta_0}{\sigma_{\text{minimum}}^2} = L^2 \vartheta_0^2.$$

Recall that $L = 1/\vartheta_0$ if \mathbf{X} follows a GPP or if \mathbf{X} is in a neighborhood of a GPP as in (6) and, thus, $\hat{\vartheta}_n$ is in this case regular and asymptotically efficient.

Corollary 4.2. *Suppose in addition to the conditions of Theorem 4.1 that \mathbf{X} is a GPP or it is in a neighborhood of a GPP as in (6). Then $\hat{\vartheta}_n = \tau(n)/(n|c_n|)$, $n \in \mathbb{N}$, is a regular estimator sequence with asymptotic minimum variance ϑ_0 within the class of regular estimators.*

A regular estimator sequence can in general be obtained as follows. Suppose that ϑ_n^* is a solution of the equation

$$P_{\vartheta_n^*}(\mathbf{X} > c_n) = \frac{\tau(n)}{n}.$$

Since $\tau(n)$ is under ϑ_0 binomial $B(n, P_{\vartheta_0}(\mathbf{X} > c_n))$ distributed, ϑ_n^* is, actually, the maximum likelihood estimator of $\kappa_0 = P_{\vartheta_0}(\mathbf{X} > c_n)$ for the family $\{B(n, \kappa) = B(n, P_{\vartheta}(\mathbf{X} > c_n)) : \vartheta \in (0, 1)\}$. We suppose consistency of the sequence ϑ_n^* , $n \in \mathbb{N}$. Then we obtain from condition (D) the expansion

$$\begin{aligned} \frac{\tau(n)}{n} &= P_{\vartheta_n^*}(\mathbf{X} > c_n) \\ &= \int_0^1 (1 + L(\vartheta_n^* - \vartheta_0) + r_{\vartheta_0}(u, \vartheta_n^*, c_n)) f_{\vartheta_0, c_n}(u) du \\ &= (1 + L(\vartheta_n^* - \vartheta_0) + o_{P_{\vartheta_0}}(|\vartheta_n^* - \vartheta_0|) + O(|c_n|^\gamma)) P_{\vartheta_0}(\mathbf{X} > c_n), \end{aligned}$$

which implies

$$(n|c_n|)^{1/2}(\vartheta_n^* - \vartheta_0) = \frac{1}{L\vartheta_0}Z_n + o_{P_{\vartheta_0}}(1).$$

As a consequence we obtain from (9) and (10) with $\vartheta_n = \vartheta_n(\xi)$

$$(n|c_n|)^{1/2}(\hat{\vartheta}_n^* - \vartheta_n) \rightarrow_{D_{\vartheta_n}} N\left(0, \frac{1}{L^2\vartheta_0^2}\right),$$

and, thus, ϑ_n^* , $n \in \mathbb{N}$, is a regular estimator sequence with asymptotic minimum variance.

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